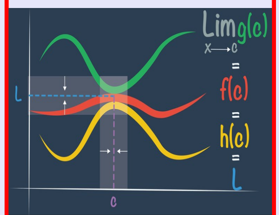
Calculus I

Lecture 4



Feb 19-8:47 AM

Class Qz 5:

Use Quadratic Formula to Solve $x^2+25=10x$.

$$ax^2+bx+c=0$$

$$x^2+25=10x$$

$$x^2+25-10x=0$$

$$x^2-10x+25=0$$

$a=1$ $\quad b=-10$ $\quad c=25$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-(-10)\pm\sqrt{(-10)^2-4(1)(25)}}{2(1)}$$

$$=\frac{10\pm\sqrt{100-100}}{2}=\frac{10\pm 0}{2}=\frac{10}{2}=\boxed{5}$$

Jan 7-12:11 PM

1) Evaluate $\lim_{x\to 4^+} \frac{4-x}{|x-4|}$

$|x| = \begin{cases} -x & \text{if } x<0 \\ x & \text{if } x>0 \end{cases}$

$|x-4| = \begin{cases} -(x-4) & \text{if } x-4<0 \\ x-4 & \text{if } x-4>0 \end{cases}$

$|x-4| = \begin{cases} -(x-4) & \text{if } x<4 \\ x-4 & \text{if } x>4 \end{cases}$

$\lim_{x\to 4^+} \frac{4-x}{|x-4|} = \lim_{x\to 4^+} \frac{\overset{-1}{\cancel{4-x}}}{\cancel{x-4}}$

$= \lim_{x\to 4^+} (-1) = \boxed{-1}$

Since $x \to 4^+$

$x>4$

4

Jan 8-8:08 AM

2) Evaluate $\lim_{x\to 3} \frac{x^3-3x^2}{\sqrt{x+6}-x} = \frac{3^3-3(3)^2}{\sqrt{3+6}-3} = \frac{27-27}{3-3} = \frac{0}{0}$

I.F.

Recall

$(A-B)(A+B) = A^2-B^2$

$= \lim_{x\to 3} \frac{x^2(x-3)(\sqrt{x+6}+x)}{(\sqrt{x+6}-x)(\sqrt{x+6}+x)}$

$= \lim_{x\to 3} \frac{x^2(x-3)(\sqrt{x+6}+x)}{(\sqrt{x+6})^2-(x)^2}$

$= \lim_{x\to 3} \frac{x^2(x-3)(\sqrt{x+6}+x)}{x+6-x^2} = \lim_{x\to 3} \frac{x^2(x-3)(\sqrt{x+6}+x)}{-(x^2-x-6)}$

$= \lim_{x\to 3} \frac{x^2\cancel{(x-3)}(\sqrt{x+6}+x)}{-\cancel{(x-3)}(x+2)} = -\lim_{x\to 3} \frac{x^2(\sqrt{x+6}+x)}{x+2}$

$= -\frac{3^2(\sqrt{3+6}+3)}{3+2} = -\frac{9\cdot 6}{5} = \boxed{-\frac{54}{5}}$

Jan 8-8:13 AM

3) Evaluate $\lim_{x\to 2} \dfrac{x^2+2x-8}{x^4-16} = \dfrac{2^2+2(2)-8}{2^4-16} = \dfrac{0}{0}$

I.F.

$A^2-B^2=(A+B)(A-B)$

$A^2+B^2 = \text{Prime}$

$$= \lim_{x\to 2} \frac{(x+4)(x-2)}{(x^2+4)(x^2-4)}$$

$$= \lim_{x\to 2} \frac{(x+4)\cancel{(x-2)}}{(x^2+4)(x+2)\cancel{(x-2)}}$$

$$= \frac{2+4}{(2^2+4)(2+2)} = \frac{6}{8\cdot 4} = \boxed{\frac{3}{16}}$$

Jan 8-8:21 AM

4) Evaluate $\lim_{x\to 1} \left[\dfrac{1}{x-1} + \dfrac{1}{x^2-3x+2}\right] = \dfrac{1}{0} + \dfrac{1}{0}$

undefined

$$= \lim_{x\to 1} \left[\frac{1(x-2)}{(x-1)(x-2)} + \frac{1}{(x-1)(x-2)}\right]$$

$$= \lim_{x\to 1} \frac{x-2+1}{(x-1)(x-2)} = \lim_{x\to 1} \frac{\cancel{x-1}^{\,1}}{\cancel{(x-1)}(x-2)}$$

$$= \lim_{x\to 1} \frac{1}{x-2}$$

$$= \frac{1}{1-2} = \boxed{-1}$$

Jan 8-8:27 AM

5) If $2x-1 \le f(x) \le x^2$ for $0<x<3$

Find $\lim\limits_{x\to1} f(x)$

Since $f(x)$ is between two other functions $\Rightarrow$ S.T.

$$\lim_{x\to1}(2x-1)=2(1)-1=1$$

$$\lim_{x\to1}x^2=1^2=1$$

By S.T.

$$\lim_{x\to1} f(x)=\boxed{1}$$

Jan 8-8:33 AM

6) Find a relationship between $\varepsilon>0$ and $\delta>0$ such that $\lim\limits_{x\to2}(14-5x)=4$.

$f(x)=14-5x$

$a=2$

$L=4$ ✓

For every $\varepsilon>0$, there is a $\delta>0$ such that $|f(x)-L|<\varepsilon$ whenever $|x-a|<\delta$

$|14-5x-4|<\varepsilon$ whenever $|x-2|<\delta$

$|-5x+10|<\varepsilon$

$|-5(x-2)|<\varepsilon$

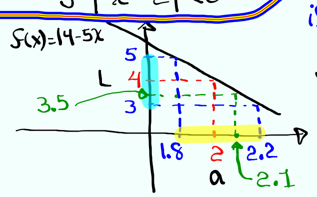
$|-5|\,|x-2|<\varepsilon$

$5\,|x-2|<\varepsilon$

Divide by 5

$|x-2|<\frac{\varepsilon}{5}$

Pick $\boxed{\delta=\frac{\varepsilon}{5}}$



if $\varepsilon=1$, $\delta=\frac{1}{5}=.2$

for $x=2.1$

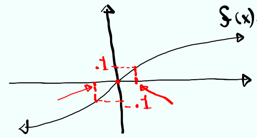
$f(2.1)=14-5(2.1)$

$=14-10.5=\boxed{3.5}$

Jan 8-8:37 AM

For $\boxed{\varepsilon = .1}$, find a $\delta > 0$ such that $\lim_{x \to 0} \sqrt[3]{x} = 0$.

$f(x) = \sqrt[3]{x}$



$f(x) = \sqrt[3]{x}$

$a = 0$

$L = 0$ ✓

For every $\varepsilon > 0$, there is a $\delta > 0$ such that

$|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$

$|\sqrt[3]{x} - 0| < .1$ whenever $|x - 0| < \delta$

$|\sqrt[3]{x}| < .1$ whenever $|x| < \delta$

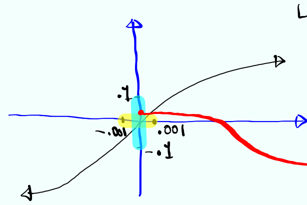
Cube both sides

$|(\sqrt[3]{x})^3| < (.1)^3$

$|x| < .001$

$|x| < .001$

Pick $\delta = .001$



Let $x = 0$

$f(0) = \sqrt[3]{0} = 0$

Let $x = .0001$

$f(.0001) = \sqrt[3]{.0001}$

$= \boxed{.046}$

Jan 8-8:47 AM

Use ε and δ to prove $\lim_{x \to 2} (x^2 - 3x) = -2$.

$f(x) = x^2 - 3x$

For every $\varepsilon > 0$, there is a $\delta > 0$ such that

$a = 2$

$L = -2$ ✓

$|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$

$|x^2 - 3x - (-2)| < \varepsilon$ whenever $|x - 2| < \delta$

$|x^2 - 3x + 2| < \varepsilon$ " $|x - 2| < \delta$

$|(x - 1)(x - 2)| < \varepsilon$

$|x - 1||x - 2| < \varepsilon$

Bound Keep

If $|x - 1| < C$, then $C|x - 2| < \varepsilon$, $|x - 2| < \frac{\varepsilon}{C}$

When working with polynomial functions, we want $\delta \le 1$

$|x - 2| < \delta \le 1$

$|x - 2| < 1$

$-1 < x - 2 < 1$

Add 1

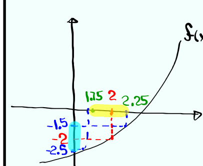
$-1 + 1 < x - 2 + 1 < 1 + 1$

$0 < x - 1 < 2$

$|x - 1| < 2$

$\boxed{C = 2}$

$\boxed{\text{We pick } \delta = \min\left\{1, \frac{\varepsilon}{2}\right\}}$



Pick $\varepsilon = .5$

$\delta = \min\left\{1, \frac{\varepsilon}{2}\right\}$

$= \min\left\{1, \frac{.5}{2}\right\}$

$= \min\{1, .25\} = .25$

If $x = 2.1$, $f(2.1) = 2.1^2 - 3(2.1) = \boxed{-1.89}$

Jan 8-8:59 AM

Show $f(x)=\frac{x-4}{x-2}$ is cont. at $x=3$.

we need to verify $\lim\limits_{x\to 3} f(x) = f(3)$

1) $f(3)=\frac{3-4}{3-2}=\frac{-1}{1}=-1$

2) $\lim\limits_{x\to 3} f(x)=\lim\limits_{x\to 3}\frac{x-4}{x-2}=\frac{3-4}{3-2}=\frac{-1}{1}=-1$

So $f(x)=\frac{x-4}{x-2}$ is cont. at $x=3$.

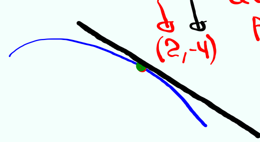
Jan 8-9:14 AM

Find slope of the tangent line to the graph of $f(x)=4x-3x^2$ at $(2,-4)$.

Polynomial
Quadratic
Parabola
opens downward

$f(2)=4(2)-3(2)^2$
$=8-12$
$=-4$

x y
$\downarrow$ $\downarrow$
$(2,-4)$



$m=\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$m=\lim\limits_{h\to 0}\frac{4(x+h)-3(x+h)^2-4x+3x^2}{h}$

$=\lim\limits_{h\to 0}\frac{\cancel{4x}+4h-\cancel{3x^2}-6xh-3h^2\cancel{-4x}\cancel{+3x^2}}{h}$

$=\lim\limits_{h\to 0}\frac{\cancel{h}(4-6x-\cancel{3h}^{\,0})}{\cancel{h}}=4-6x$

Need to evaluate at $x=2$
tangent pt

$m=4-6(2)$
$=4-12=\boxed{-8}$

Equation of tan. line

$y-y_1=m(x-x_1)$

$y-(-4)=-8(x-2)$

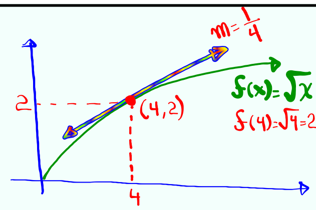
$\boxed{y=-8x+12}$

Jan 8-9:18 AM

Given $f(x) = \sqrt{x}$

1) Graph $f(x)$

2) label the point with $x = 4$



3) Draw the tan. line to the graph at the point with $x = 4$.

4) Find the slope of this tangent line.

$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \quad \boxed{\text{Evaluate at } x = 4.}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{\cancel{x} + \cancel{h} - \cancel{x}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \quad \text{for } x = 4 \to m = \frac{1}{2\sqrt{4}} = \boxed{\frac{1}{4}}$$

5) find the equation of that tangent line.

$$y - y_1 = m(x - x_1)$$

$$y - 2 = \frac{1}{4}(x - 4)$$

$$y - 2 = \frac{1}{4}x - 1 \to \boxed{y = \frac{1}{4}x + 1}$$

Jan 8-9:29 AM

Given $f(x) = \frac{2x+1}{x+2}$ $\quad f(1) = \frac{2(1)+1}{1+2} = \frac{3}{3} = 1$

1) Domain $\quad x + 2 \neq 0$

$x \neq -2$

-2

$(-\infty, -2) \cup (-2, \infty)$

2) Y-Int $\quad x = 0$

$f(0) = \frac{2(0)+1}{0+2} = \frac{1}{2} \to (0, \frac{1}{2})$

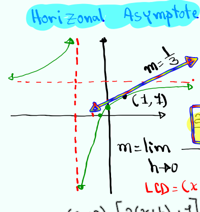
3) x-Int $\quad y = 0 \quad f(x) = 0 \quad \frac{2x+1}{x+2} = 0 \quad 2x+1 = 0$

$x = -\frac{1}{2}$

$\to (-\frac{1}{2}, 0)$

4) Vertical Asymptote at $x = -2$

Horizonal Asymptote at $y = 2$



$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

evaluate at $x = 1$

$$m = \lim_{h \to 0} \frac{\boxed{\frac{2(x+h)+1}{x+h+2}} - \boxed{\frac{2x+1}{x+2}}}{h}$$

LCD $= (x+h+2)(x+2)$

$$= \lim_{h \to 0} \frac{(x+2) \cdot [2(x+h)+1] - (x+h+2) \cdot (2x+1)}{h(x+h+2)(x+2)}$$

$$= \lim_{h \to 0} \frac{(x+2)(2x+2h+1) - (x+h+2)(2x+1)}{h(x+h+2)(x+2)}$$

$$= \lim_{h \to 0} \frac{\cancel{2x^2} + \cancel{2xh} + \cancel{x} + \cancel{4x} + 4h + \cancel{2} - \cancel{2x^2} - \cancel{x} - \cancel{2hx} - h - \cancel{4x} - \cancel{2}}{h(x+h+2)(x+2)}$$

$$= \lim_{h \to 0} \frac{3\cancel{h}}{\cancel{h}(x+h+2)(x+2)} = \lim_{h \to 0} \frac{3}{(x+\cancel{h}+2)(x+2)} = \frac{3}{(x+2)^2}$$

at $x = 1$

$$m = \frac{3}{(1+2)^2} = \frac{3}{9} = \boxed{\frac{1}{3}}$$

Eqn. of tan. line

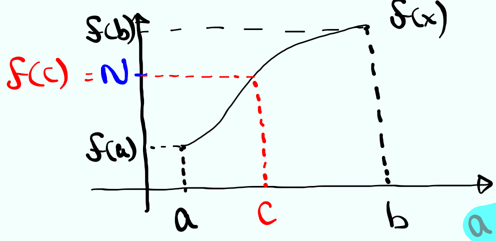
$$y - 1 = \frac{1}{3}(x - 1)$$

$$\ldots \boxed{y = \frac{1}{3}x + \frac{2}{3}}$$

Jan 8-10:05 AM

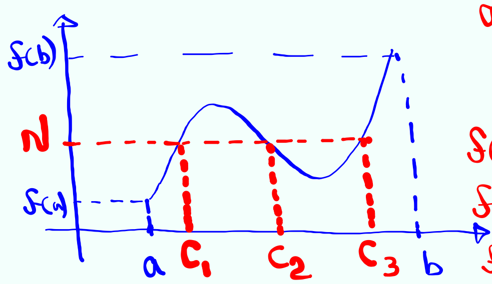
Intermediate Value Theorem:

Suppose $f(x)$ is a cont. function for all value $a \le x \le b$ and $f(a) \ne f(b)$



Suppose N is a number between $f(a)$ & $f(b)$

There is at least a number c between a & b such that $f(c) = N$



$f(c_1) = N$

$f(c_2) = N$

$f(c_3) = N$

Jan 8-10:25 AM

Show $4x^3 - 6x^2 + 3x - 2 = 0$ has a Solution between 1 and 2.

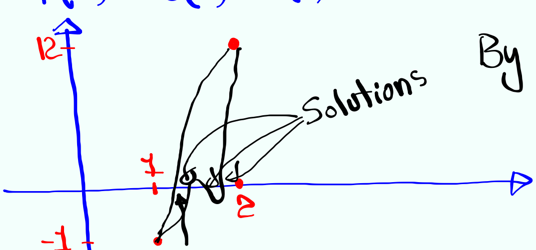
Let $f(x) = 4x^3 - 6x^2 + 3x - 2$

Polynomial function $\rightarrow$ Continuous everywhere

$f(1) = 4(1)^3 - 6(1)^2 + 3(1) - 2 = 4 - 6 + 3 - 2 = -1$

$f(2) = 4(2)^3 - 6(2)^2 + 3(2) - 2 = 32 - 24 + 6 - 2 = 12$

By I.V.T.



Solution $f(x) = 0$

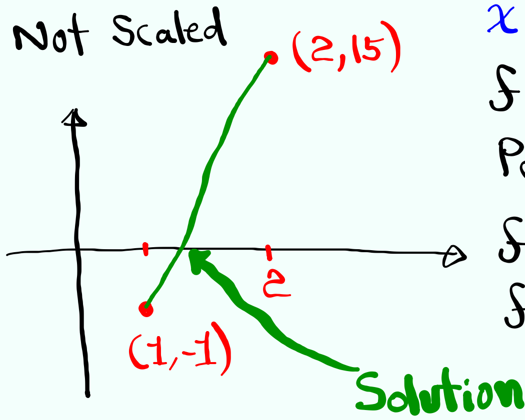
Jan 8-10:33 AM

Show $x^4 + x = 3$ over $[1,2]$.

Method I: Move 3 to the left side

$x^4 + x - 3 = 0$

Not Scaled



$f(x) = x^4 + x - 3$

Polynomial, Cont. $(-\infty, \infty)$

$f(1) = -1$

$f(2) = 15$

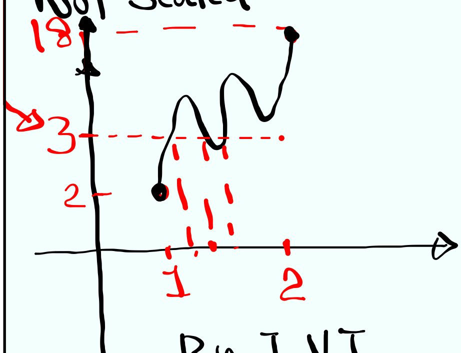
by I.V.T., $f(x)$ has to be equal to 0 at least once between 1 & 2.

Jan 8-10:39 AM

Method II

$x^4 + x = \boxed{3}$

Not Scaled



By I.V.T.

$f(x) = x^4 + x$, $[1,2]$

Polynomial

Continuous $(-\infty, \infty)$

$f(1) = 1^4 + 1 = 2$

$f(2) = 2^4 + 2 = 18$

Jan 8-10:43 AM

Show there is Solution on $[0,1]$ for

$\cos x = x$

$\underbrace{\cos x}_{\text{Cont. } (-\infty,\infty)} - \underbrace{x}_{\text{Cont. } (-\infty,\infty)} = 0$

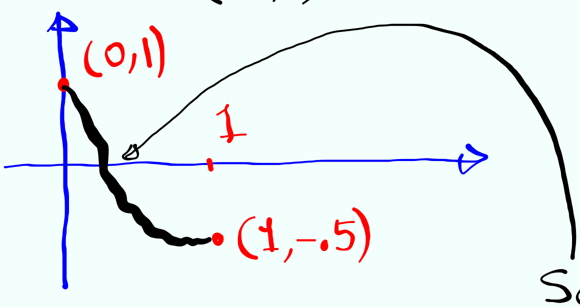
$F(x) = \cos x - x$

is cont. $(-\infty,\infty)$

$F(0) = \cos 0 - 0 = \boxed{1}$

$f(1) = \cos 1 - 1 \approx \boxed{-.5}$

by I.V.T.



Jan 8-10:48 AM

Show $\underbrace{x^{10} - 10x^2 + 5}$ $= 0$ has a root in $[0,2]$

From PreCalculus or College Algebra at most 10 roots.

Some real roots and Some imaginary roots (maybe)

$f(x) = x^{10} - 10x^2 + 5$

Polynomial, Cont. $(-\infty,\infty)$

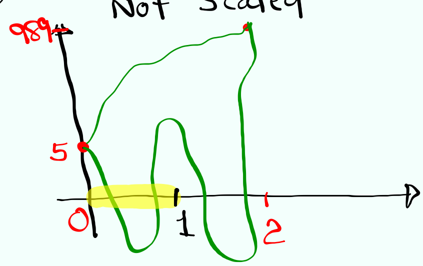
$f(0) = 5$, $f(2) = 989$

Pick $[0,1]$

$f(0) = 5$

$f(1) = -4$

by I.V.T, $f(x) = 0$ at least once in $[0,1]$



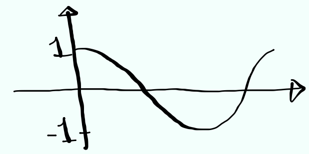
Jan 8-10:54 AM

Evaluate $\lim_{x\to 0} x^2\cos\frac{1}{x^2} = 0^2\cdot\cos\boxed{\frac{1}{0^2}}$

$= 0\cos\frac{1}{0}$

undefined

Recall $-1\le \cos x \le 1$



$-1\le \cos\frac{1}{x^2} \le 1$

Multiply by x^2

$-1\cdot x^2 \le x^2\cdot\cos\frac{1}{x^2} \le 1\cdot x^2$

$\boxed{-x^2} \le x^2\cos\frac{1}{x^2} \le \boxed{x^2}$

$\lim_{x\to 0} x^2 = 0$

$\lim_{x\to 0} -x^2 = 0$

$\lim_{x\to 0} x^2\cos\frac{1}{x^2} = 0$

By S.T.

Jan 8-11:02 AM

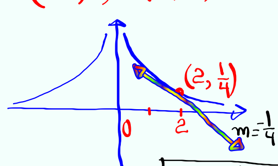
$f(x) = \frac{1}{x^2}$ $\quad f(2) = \frac{1}{2^2} = \frac{1}{4}$

1) Domain $x\neq 0$

$(-\infty,0)\cup(0,\infty)$

2) Range $y>0$

$(0,\infty)$



$\lim_{x\to 0^+} f(x) = \infty$

$\lim_{x\to 0^-} f(x) = \infty$

$\lim_{x\to\infty} f(x) = 0$

$y - \frac{1}{4} = -\frac{1}{4}(x-2)$

$\boxed{y =}$

$\lim_{x\to 0} f(x) = \infty$

$f(0)$ undefined

$\lim_{x\to -\infty} f(x) = 0$

$m = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\boxed{\frac{1}{(x+h)^2}} - \boxed{\frac{1}{x^2}}}{h}$

then evaluate at $x=2$.

LCD $= (x+h)^2\cdot x^2$

$= \lim_{h\to 0}\frac{x^2 - (x+h)^2}{h(x+h)^2x^2} = \lim_{h\to 0}\frac{\cancel{x^2} - \cancel{x^2} - 2xh - h^2}{h(x+h)^2x^2}$

$= \lim_{h\to 0}\frac{\cancel{h}(-2x - h^{\to 0})}{\cancel{h}(x+h^{\to 0})^2x^2} = \frac{-2x}{x^2\cdot x^2} = \frac{-2}{x^3}$

at $x=2$ $\quad m = \frac{-2}{2^3} = \frac{-2}{8} = \boxed{-\frac{1}{4}}$

Jan 8-11:08 AM

Find $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ for $f(x)=ax^3$.

$$\lim_{h\to 0} \frac{a(x+h)^3 - ax^3}{h}$$

$$= \lim_{h\to 0} \frac{a(x^3+3x^2h+3xh^2+h^3) - ax^3}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{ax^3} + 3ax^2h + 3axh^2 + ah^3 \cancel{-ax^3}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(3ax^2 + 3axh + ah^2)}{\cancel{h}} = \boxed{3ax^2}$$

Jan 8-11:24 AM

Evaluate $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ for $f(x)=\frac{x}{x+2}$.

$$= \lim_{h\to 0} \frac{\boxed{\frac{x+h}{x+h+2}} - \boxed{\frac{x}{x+2}}}{h} = \lim_{h\to 0} \frac{(x+2)(x+h) - x(x+h+2)}{h(x+h+2)(x+2)}$$

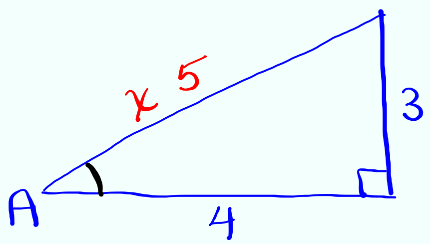
LCD $= (x+h+2)(x+2)$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + \cancel{xh} + \cancel{2x} + 2h \cancel{-x^2} \cancel{-xh} \cancel{-2x}}{h(x+h+2)(x+2)}$$

$$= \lim_{h\to 0} \frac{2}{(x+\cancel{h}+2)(x+2)} = \frac{2}{(x+2)^2}$$

Jan 8-11:30 AM

Consider the right triangle below



1) Find the hypotenuse.

$3^2 + 4^2 = x^2$

$\boxed{x = 5}$

2) $\sin A = \frac{3}{5}$ $\qquad \csc A = \frac{5}{3}$

$\cos A = \frac{4}{5}$ $\qquad \sec A = \frac{5}{4}$

$\tan A = \frac{3}{4}$ $\qquad \cot A = \frac{4}{3}$

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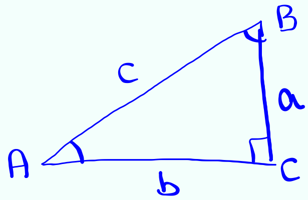
Use the right triangle below to prove



1) $\sin^2 A + \cos^2 A = 1.$

$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 =$

$\frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{a^2+b^2}{c^2}$

$= \frac{c^2}{c^2} = \boxed{1}$

2) $1 + \tan^2 A = \sec^2 A$

$1 + \left(\frac{a}{b}\right)^2$

$1 + \frac{a^2}{b^2} = \frac{b^2}{b^2} + \frac{a^2}{b^2} = \frac{b^2+a^2}{b^2} = \frac{c^2}{b^2} = \left(\frac{c}{b}\right)^2$

$= \boxed{\sec^2 A}$

$\cos A = \frac{b}{c}$

$\sec A = \frac{c}{b}$

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Open notes:

Class QZ 6

Warning!

Notation Matters

Evaluate $\lim_{x \to 0} \frac{1-\sqrt{1-x^2}}{x} = \frac{1-\sqrt{1-0^2}}{0} = \frac{1-1}{0} = \frac{0}{0}$ I.F.

$$= \lim_{x \to 0} \frac{(1-\sqrt{1-x^2})(1+\sqrt{1-x^2})}{x(1+\sqrt{1-x^2})} = \lim_{x \to 0} \frac{1-(1-x^2)}{x(1+\sqrt{1-x^2})}$$

$$= \lim_{x \to 0} \frac{\cancel{1} \cancel{-1} + x^{\cancel{2}}}{\cancel{x}(1+\sqrt{1-x^2})} = \lim_{x \to 0} \frac{x}{1+\sqrt{1-x^2}}$$

$$= \frac{0}{1+\sqrt{1-0^2}} = \frac{0}{2} = \boxed{0}$$

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